Objective

In this experiment you will use Euler’s method to solve some problems involving the motion of a projectile subject to air resistance.

Apparatus

Microsoft Excel.

Theory

Euler’s method is a numerical method of approximating the solution of an ordinary differential equation (ode) at discrete points in some interval \((x_0, x_1, \ldots, x_n)\).

Consider the ode

\[
\frac{dy}{dx} = f'(x)
\]

which has solution

\[
y = f(x)
\]

The approximation

\[
\frac{dy}{dx} \approx \frac{\Delta y}{\Delta x}
\]

means that Equation 1 can be written

\[
\Delta y = f'(x)\Delta x
\]

If \(x_0\) and \(y_0\) (the initial conditions) are known then

\[
x_1 = x_0 + \Delta x
\]
\[
y_1 = y_0 + f'(x_0)\Delta x
\]

The process is then be repeated; the general algorithm is

\[
x_n = x_{n-1} + \Delta x
\]
\[
y_n = y_{n-1} + f'(x_{n-1})\Delta x
\]

Example

The ode

\[
\frac{dy}{dx} = 2x
\]

has solution

\[
y = x^2
\]

with \(x_0 = y_0 = 0\). We will integrate the ode in the interval \([0,1]\) using Euler’s method with \(\Delta x = 0.1\).

Iterate at the first step

\[
x_1 = 0 + 0.1
\]
\[
y_1 = 0 + 2(0)(0.1)
\]

the second

\[
x_2 = 0.1 + 0.1
\]
\[
y_2 = 0 + 2(0.1)(0.1)
\]
and continue to $x = 1.0$. The results are shown below along with the actual value of the function at each point.

<table>
<thead>
<tr>
<th>Step</th>
<th>$x$</th>
<th>$y_{Euler}$</th>
<th>$y_{Actual}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>0.1</td>
<td>0.01</td>
<td>0.01</td>
</tr>
<tr>
<td>2</td>
<td>0.2</td>
<td>0.04</td>
<td>0.04</td>
</tr>
<tr>
<td>3</td>
<td>0.3</td>
<td>0.09</td>
<td>0.09</td>
</tr>
<tr>
<td>4</td>
<td>0.4</td>
<td>0.16</td>
<td>0.16</td>
</tr>
<tr>
<td>5</td>
<td>0.5</td>
<td>0.25</td>
<td>0.25</td>
</tr>
<tr>
<td>6</td>
<td>0.6</td>
<td>0.36</td>
<td>0.36</td>
</tr>
<tr>
<td>7</td>
<td>0.7</td>
<td>0.49</td>
<td>0.49</td>
</tr>
<tr>
<td>8</td>
<td>0.8</td>
<td>0.64</td>
<td>0.64</td>
</tr>
<tr>
<td>9</td>
<td>0.9</td>
<td>0.81</td>
<td>0.81</td>
</tr>
<tr>
<td>10</td>
<td>1.0</td>
<td>1.00</td>
<td>1.00</td>
</tr>
</tbody>
</table>

The error in the values of $y$ at $x = 1.0$ is about 10%. However, with $x = 0.01$ the error is only 1.0% (0.10% with $x = 0.001$). You can see the benefit of a small step size.

Physics is all about rates of change (ode’s) so this method can be used to solve a wide variety of physics problems.

Remember that

\[ a = \frac{dv}{dt} \quad \text{and} \quad v = \frac{dx}{dt} \]

and consider an object moving in one dimension subject to a constant rate of acceleration $a$.

If it starts from rest ($x_o = v_o = t_o = 0$) then

\[
\begin{align*}
t_1 &= 0 + \Delta t \\
x_1 &= 0 + 0 \Delta t \\
v_1 &= 0 + a \Delta t \\
a_1 &= a
\end{align*}
\]

and in general

\[
\begin{align*}
t_n &= t_{n-1} + \Delta t \\
x_n &= x_{n-1} + v_{n-1} \Delta t \\
v_n &= v_{n-1} + a_{n-1} \Delta t \\
a_n &= a
\end{align*}
\]

This is remarkably easy to implement in a spreadsheet such as Excel.

The first 10 steps of the problem are shown below with $a = 0.2\text{m/s}^2$ (cell G2). Here, the formulas are displayed in the cells rather than the results to better demonstrate the process [you can check the solutions analytically].

Note that the formulas in Row 3 were copied down the rest of the way. This is the beauty of Excel - with judicious use of relative and absolute cell references the first iterative row can be copied down as far as needed - which may be several thousand steps.

Although the method will work on any ode it is most often employed on those difficult or impossible to solve analytically.

The acceleration of the object need not be constant; it could be a function of any number of variables - position, velocity, or time.

As long as initial values can be quantified it can be used. But, remember - all numerical methods give but approximate solutions and are subject to accumulation error. You can minimize this with very small $\Delta t$.

### Air Resistance on a Projectile

Assume that air resistance for a projectile is proportional to the square of the projectile velocity; i.e., the drag force on the projectile is given by

\[ f = Dv^2 \]

where

\[ v^2 = v_x^2 + v_y^2 \]

and

\[ D = \frac{\rho CA}{2} \]

Here, $\rho$ is the density of air, $C$ a dimensionless constant called the drag coefficient (0.2 to 1.0 for baseballs, tennis balls, etc.), and $A$ the cross-sectional area of the projectile.
At any point in the motion of the projectile, the drag force is directed opposite to the direction of the velocity, as shown below.

Using Newton’s second law in each dimension yields

\[-Dv^2 \cos \theta = -Dv(v \cos \theta) = ma_x\]
\[-mg - Dv^2 \sin \theta = -mg - Dv(v \sin \theta) = ma_y\]

thus the corresponding accelerations are just

\[a_x = -\left( \frac{D}{m} \right) vv_x\]
\[a_y = -g - \left( \frac{D}{m} \right) vv_y\]

From an Euler method standpoint this would be implemented as

\[a_{nx} = -\left( \frac{D}{m} \right) v_n v_{nx}\] \hspace{1cm} (4)
\[a_{ny} = -g - \left( \frac{D}{m} \right) v_n v_{ny}\] \hspace{1cm} (5)

Obviously the \(x\) and \(y\) dimensions must be solved separately. If you know the initial values \(x_0, y_0, v_{x_0}\), and \(v_{y_0}\) then the method can be used.

Problems

Each of the problems below should be solved in Excel using Euler’s method.

One of the advantages of any spreadsheet (such as Excel) is that they are dynamic - recalculating every time a value is changed that affects other values.

Since every problem involves a projectile subject to the the same acceleration (Equations 4 and 5) - you need to set up only one worksheet of the workbook with an Euler table.

Proceed as follows: at the top of the worksheet place the basic parameters \(g, r, C, \rho, m, v_o, \theta\) (in degrees), and \(\Delta t\). Add the calculated parameters \(A\) and \(D\).

Below this start the table of Euler calculations; it should contain 8 columns - Step, \(t\), \(x\), \(v_x\), \(a_x\), \(y\), \(v_y\), and \(a_y\). The first row (Step 0) is initial values for all quantities.

Subsequent rows are determined with the Euler equations. If you reference the parameters at the top of the worksheet in your equations you need type them in only once, for Step 1; they can then be copied down as far as you need to go.

Remember that the trigonometric functions in Excel take their arguments in radians. You have the angle above the table in degrees for simplicity but you can use the RADIANS() function in your equations to convert this value.

In addition, this same table can be used to solve all of the problems just by changing the parameters and/or initial values at the top. Each time, just scan your table to obtain the desired answer.

At the top of the worksheet to the right of the parameters and Euler table list the answers you obtained for each problem along with the \(\Delta t\) used in each of them.

Before you email your Excel file, delete all but about 25 rows of the Euler table.

1. As a test program find the trajectory of a baseball with and without air resistance (for none, set the parameter \(C = 0\) which makes \(D = 0\)). The radius of the baseball is 0.0366m, and \(A = \pi r^2\).

   Its mass is 0.145kg, the drag coefficient is 0.5, and the density of air 1.2kg/m\(^3\). The baseball is given an initial velocity of 50m/s at an angle of 35\(^\circ\) above the +x-axis. With drag the range of the baseball is just over 100m; without drag the range is almost 240m.

2. The home field for the Colorado Rockies is in Denver (altitude 1 mile), where the air density is only 1.0kg/m\(^3\). How much farther will the baseball in the trial program go compared to sea level?

3. Estimate the maximum distance a human being can throw a ping-pong ball. A ping-pong ball has a radius of 0.019m and a mass of 0.0024kg. Use a drag coefficient of 0.5 and a density of 1.2kg/m\(^3\).

   The fastest pitchers can throw at about 100mi/h. Why can a baseball be thrown much farther than
a ping-pong ball?

4. In tennis, 100mi/h is a very good serving speed. How fast is the ball moving when it crosses the baseline in the opposite court, about 24m distant? Use a mass of 0.055kg, a radius of 0.031m, a drag coefficient of 0.75, and a density of 1.2kg/m³. The ball leaves the racquet moving horizontally and does not hit the ground until after it crosses the opposite court baseline.

5. Decide whether or not air resistance really is negligible in a typical Projectile Motion lab experiment. The projectile sphere has a mass of ≈16g, a diameter of ≈1.5cm, a small C (i.e., closer to 0.2 than 1), and an initial velocity of ≈5.00m/s. For a launch angle of 45°, examine the motion in both cases – drag and no drag. What is the ratio

\[
\frac{\text{range}_{\text{drag}}}{\text{range}}
\]

Is air resistance negligible or not?